

HOPF-HOCHSCHILD (CO)HOMOLOGY OF MODULE ALGEBRAS

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1. INTRODUCTION

Our goal in this paper is to define a version of Hochschild homology and cohomology suitable for a class of algebras admitting compatible actions of bialgebras, called “module algebras” (Definition 2.1). Our motivation lies in the following problem: for an algebra A which admits a module structure over an arbitrary bialgebra B compatible with its product structure, the Hochschild or the cyclic bicomplexes associated with this algebra need not be differential graded B -modules. The obstruction which prevents these complexes from being B -linear is trivial whenever the bialgebra B is cocommutative, as in the case of group rings and universal enveloping algebras. Yet the same obstruction is far from being trivial if the underlying bialgebra is non-cocommutative. In the sequel, we will investigate how much of the Hochschild homology is retained after dividing this obstruction out. To this end, we will construct a new differential graded B -module $\mathrm{QCH}_*(A, B, V)$ (Proposition 2.10 and Definition 2.11) for a B -module algebra A and a B -equivariant A -bimodule V (Definition 2.2). We will define $HH_*^{\mathrm{Hopf}}(A, B, V)$ the Hopf-Hochschild homology of A with coefficients in V as the homology of the complex $k \otimes_B \mathrm{QCH}_*(A, B, V)$. We would like to point out that the same strategy worked remarkably well in the case of cyclic cohomology of module coalgebras. In [13] we show that if we start with the cocyclic bicomplex of a module coalgebra twisted by a stable anti-Yetter-Drinfeld module, dividing the analogous obstruction results in the Hopf cyclic complex of [10] which was an extension of the Hopf cyclic cohomology of Connes and Moscovici [7].

In the context of cyclic (co)homology and K -Theory, one of the most commonly used tools dealing with module algebras has been “crossed product algebras” (Definition 3.1). There is a large body of work dealing with algebras admitting actions of discrete groups and compact Lie groups, e.g. [9, 11, 2, 5, 12, 8, 4] and references therein, which utilizes this tool to its fullest extent. Also, there have been successful attempts in defining equivariant cyclic (co)homology and K -Theory for module algebras over Hopf algebras [3, 1, 17] again by using crossed product algebras. Crossed product algebras enter in our picture in Corollary 4.4 where we show that Hopf-Hochschild homology can also be defined as a derived functor on the category of representations of a crossed product algebra.

The last result we prove is the Morita invariance of the Hopf-Hochschild homology and cohomology (Theorem 7.9 and Theorem 8.4). Our proof utilizes some additional tools from functor homology [16, 18]. In doing so, we observe that the category of representations of a crossed product algebra is rather small

for computing equivariant invariants. However, the short-comings of this category can be overcome by using “ B -categories” (Definition 6.1). We refer the reader to Remark 6.9 for a more detailed analysis on this subject.

Here is the plan of this paper: In Section 2 we give the basic definition of Hopf–Hochschild complex of module algebras with coefficients in equivariant bimodules. We also point out the connections between Hopf–Hochschild homology and Hopf cyclic cohomology (Remark 2.13). In Section 3 we define Hopf–Hochschild cohomology of a B -module algebra and calculate it for lower dimensions. We also give a derived functor interpretation of the Hopf–Hochschild cohomology in terms of crossed product algebras. In Section 4 we extend the derived functor interpretation to Hopf–Hochschild homology. Section 5 and Section 6 contain technical results needed toward proving Morita invariance of Hopf–Hochschild (co)homology in Section 7 in its full generality. In Section 8 we develop a generalized twisting method for coefficient bifunctors or bimodules by using Yetter–Drinfeld modules similar to the method of twisting developed in [10]. In this last Section, we also prove Morita invariance for twisted Hopf–Hochschild (co)homology.

Throughout this paper, we assume k is an arbitrary field and B is an associative/coassociative unital/counital bialgebra, or a Hopf algebra with an invertible antipode whenever it is necessary. All tensor products are taken over k unless it is stated otherwise explicitly.

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2. HOPF–HOCHSCHILD HOMOLOGY

Definition 2.1. An algebra A is called a left B -module algebra if A is a B -module and

$$b(a_1 a_2) = b_{(1)}(a_1) b_{(2)}(a_2)$$

for any $b \in B$ and $a_1, a_2 \in A$. If A is unital, we also assume $b(1_A) = \varepsilon(b)1_A$ where ε is the counit of B .

Definition 2.2. Let A be a B -module algebra. An A -module V is called a B -equivariant A -module if V is both an A -module and B -module and one also has

$$b(av) = b_{(1)}(a) b_{(2)}(v)$$

for any $a \in A$ and $b \in B$.

Example 2.3. Let $B = k[G]$ be the group algebra of a discrete group G . Then an algebra A is a $k[G]$ -module algebra iff it is a G -algebra.

Example 2.4. Let $B = U(\mathfrak{g})$ is the universal enveloping algebra of Lie algebra \mathfrak{g} . Then an algebra A is a $U(\mathfrak{g})$ -module algebra iff it admits an action of \mathfrak{g} by derivations.

Example 2.5. Let B be an arbitrary Hopf algebra and A be an algebra which is also a B -bimodule. Then there is a natural action of B on A called the adjoint action which makes A into a B -module algebra and any A -module which is also a B -module V into a B -equivariant A -module. The adjoint action is defined as

$$ad_b(a) = b_{(1)}aS(b_{(2)})$$

for any $b \in B$ and $a \in A$. One can easily see that

$$ad_b(a_1a_2) = b_{(1)}a_1a_2S(b_{(2)}) = b_{(1)}a_1S(b_{(2)})b_{(3)}a_2S(b_{(4)}) = ad_{b_{(1)}}(a_1)ad_{b_{(2)}}(a_2)$$

for any $b \in B$ and $a_1, a_2 \in A$. Similarly

$$b(av) = b_{(1)}aS(b_{(2)})b_{(3)}v = ad_{b_{(1)}}(a)(b_{(2)}v)$$

for any $a \in A$, $b \in B$ and $v \in V$.

Definition 2.6. Given an algebra A and a A -bimodule V , we will use the notation $\text{CH}_*(A, V)$ to denote graded module $\bigoplus_{n \geq 0} A^{\otimes n} \otimes V$ with structure morphisms

$$\partial_j(a_1 \otimes \cdots \otimes a_n \otimes v) = \begin{cases} (\cdots \otimes a_{j+1}a_{j+2} \otimes \cdots \otimes v) & \text{if } 0 \leq j < n-1 \\ (a_1 \otimes \cdots \otimes a_nv) & \text{if } j = n-1 \\ (a_2 \otimes \cdots \otimes a_n \otimes va_1) & \text{if } j = n \end{cases}$$

which makes $\text{CH}_*(A, V)$ into a pre-simplicial module. The differential graded module with the differentials

$$d_n^{\text{CH}} = \sum_{j=0}^n (-1)^j \partial_j$$

corresponding to this pre-simplicial module is also denoted by $\text{CH}_*(A, V)$, and is called the Hochschild complex of A with coefficients in the A -bimodule V .

From this point on, we will assume A is a B -module algebra and V is a B -equivariant A -bimodule unless it is stated otherwise explicitly.

Remark 2.7. B as an algebra acts on $\text{CH}_*(A, V)$ diagonally as

$$L_b(a_1 \otimes \cdots \otimes a_n \otimes v) = b_{(1)}(a_1) \otimes \cdots \otimes b_{(n)}(a_n) \otimes b_{(n+1)}(v)$$

which makes $\text{CH}_*(A, V)$ into a graded B -module but NOT a differential graded B -module since

$$\begin{aligned} \partial_n L_b(a_1 \otimes \cdots \otimes a_n \otimes v) &= b_{(2)}(a_2) \otimes \cdots \otimes b_{(n)}(a_n) \otimes b_{(n+1)}(v)b_{(1)}(a_1) \\ &\neq b_{(1)}(a_2) \otimes \cdots \otimes b_{(n-1)}(a_n) \otimes b_{(n)}(v)b_{(n+1)}(a_1) \\ &= L_b \partial_n(a_1 \otimes \cdots \otimes a_n \otimes v) \end{aligned}$$

unless $b \in \ker(\delta)$ where $\delta = (id_2 - \tau_2)\Delta$. This means, although the B -structure on A does extend to a graded B -module structure on the ordinary Hochschild complex $\text{CH}_*(A, V)$, it extends to a differential

graded B -module structure on $\mathrm{CH}_*(A, V)$ if B is cocommutative, for instance when B is a group ring or a universal enveloping algebra. The obstruction which prevents $\mathrm{CH}_*(A, V)$ from being a differential graded B -module is the subcomplex generated by images of the commutators $[L_x, d_*^{\mathrm{CH}}]$ where L_x is the k -linear endomorphism of $\mathrm{CH}_*(A, V)$ coming from the diagonal action of $x \in B$ on $\mathrm{CH}_*(A, V)$. Now one can ask the following question: what happens if we force these differential graded k -modules to become differential graded B -modules by dividing out this obstruction? This is what we are going to do with Definition 2.9 and Proposition 2.10 for the ordinary Hochschild complex. In the sequel, we investigate homological consequences of this operation.

Let $(\mathcal{C}_*, d_*^{\mathcal{C}})$ be a differential graded k -module and let $n \in \mathbb{N}$. Then we define $\mathcal{C}_*[+n]$ the n -fold suspension of \mathcal{C}_* as the differential graded k -module $\mathcal{C}_m[+n] = \mathcal{C}_{m+n}$ with differentials $d_m^{\mathcal{C}}[+n] = d_{m+n}^{\mathcal{C}}$ for any $m \in \mathbb{Z}$. One can similarly define $\mathcal{C}_*[-n]$ for any $n \in \mathbb{N}$. Note that $H_{m \pm n}(\mathcal{C}_*[\pm n]) = H_m(\mathcal{C}_*)$.

Lemma 2.8. *For any $b \in B$, there is a morphism of differential graded k -modules of the form*

$$\mathrm{CH}_*(A, V)[+1] \xrightarrow{[L_b, \partial_{*+1}]} \mathrm{CH}_*(A, V)$$

Moreover, $[L_b, \partial_{*+1}]$ is null-homotopic for any $b \in B$.

Proof. For any $b \in B$ we consider

$$\begin{aligned} \partial_j[L_b, \partial_{n+1}](\mathbf{a} \otimes v) &= -[L_b, \partial_j]\partial_{n+1}(\mathbf{a} \otimes v) + [L_b, \partial_j\partial_{n+1}](\mathbf{a} \otimes v) \\ &= \begin{cases} [L_b, \partial_n\partial_j](\mathbf{a} \otimes v) & \text{if } 0 \leq j \leq n-1 \\ -[L_b, \partial_n]\partial_{n+1}(\mathbf{a} \otimes v) + [L_b, \partial_n\partial_n](\mathbf{a} \otimes v) & \text{if } j = n \end{cases} \\ &= \begin{cases} [L_b, \partial_n]\partial_j(\mathbf{a} \otimes v) & \text{if } 0 \leq j \leq n-1 \\ -[L_b, \partial_n]\partial_{n+1}(\mathbf{a} \otimes v) + [L_b, \partial_n]\partial_n(\mathbf{a} \otimes v) & \text{if } j = n \end{cases} \end{aligned}$$

by using the fact that $[L_b, \partial_j](\mathbf{a} \otimes v) = 0$ for any $0 \leq n, 0 \leq j \leq n-1$ and for any $(\mathbf{a} \otimes v)$ from $\mathrm{CH}_n(A, V)$.

This result immediately implies

$$d_n^{\mathrm{CH}}[L_b, \partial_{n+1}] = [L_b, \partial_n]d_n^{\mathrm{CH}}$$

The null-homotopy $\mathrm{CH}_n(A, V) \xrightarrow{s_n} \mathrm{CH}_{n+1}(A, V)[+1]$ is given by $s_n = (-1)^{n-1}L_b$ for any $n \geq 0$. \square

Definition 2.9. We define a graded B -submodule of $\mathrm{CH}_*(A, V)$ as

$$J_*(A, B, V) = \sum_{b \in B} \mathrm{im}([L_b, \partial_{*+1}])$$

Proposition 2.10. *We define a new graded B -module $\mathrm{QCH}_*(A, B, V)$ as the quotient graded B -module*

$$\mathrm{QCH}_*(A, B, V) := \mathrm{CH}_*(A, V) / J_*(A, B, V)$$

Then $\mathrm{QCH}_(A, B, V)$ is also a differential graded B -module.*

Proof. Since each $im([L_b, \partial_{*+1}])$ is a differential graded k -submodule of $CH_*(A, V)$, the submodule $J_*(A, B, V)$ is also a differential graded k -submodule of $CH_*(A, V)$. This fact implies $QCH_*(A, B, V)$ is too a differential graded k -module. Moreover, $J_*(A, B, V)$ is a graded B -submodule of $CH_*(A, V)$ since

$$L_x[L_b, \partial_{n+1}](v \otimes \mathbf{a}) = -[L_x, \partial_{n+1}]L_b(v \otimes \mathbf{a}) + [L_{xb}, \partial_{n+1}](v \otimes \mathbf{a})$$

for any $x, b \in B$, $n \geq 0$ and $(v \otimes \mathbf{a})$ from $CH_n(A, V)[+1]$. In order $QCH_*(A, B, V)$ be a differential graded B -module, we must show that $[L_b, d_*^{CH}] \equiv 0$ on $QCH_*(A, B, V)$ for any $b \in B$. This is equivalent to saying that

$$[L_b, d_n^{CH}](v \otimes \mathbf{a}) = (-1)^n [L_b, \partial_n](v \otimes \mathbf{a})$$

must be in $J_{n-1}(B, B, V)$ for any $(v \otimes \mathbf{a})$ from $CH_n(A, V)$ and for any $b \in B$ is in $J_{n-1}(A, B, V)$ which is true by definition. \square

Definition 2.11. Assume B is a bialgebra. For a B -module algebra A and a B -equivariant A -module V , we define Hopf-Hochschild homology of A with coefficients in V as the homology of the differential graded k -module ${}_B QCH_*(A, B, V)$ which is defined as $k \otimes_B QCH_*(A, B, V)$. In other words,

$$HH_n^{\text{Hopf}}(A, V) := H_n({}_B QCH_*(A, B, V))$$

for any $n \geq 0$.

Remark 2.12. Observe that if B is cocommutative, the differential graded k -module $QCH_*(A, B, V)$ is equal to the ordinary Hochschild complex $CH_*(A, V)$. In an (un)likely case when B is both cocommutative and semi-simple (such as $B = k[G]$ where G is a finite group and $char(k)$ does not divide $|G|$) then one has an isomorphism of the form $HH_*^{\text{Hopf}}(A, V) \cong k \otimes_B HH_*(A, V)$. For example, if $B = k$ the Hopf-Hochschild homology is the same as the ordinary Hochschild homology.

Remark 2.13. Assume A is an associative, but not necessarily unital k -algebra. Apart from the ordinary Hochschild complex of A , there are several other different differential graded k -modules one can associate with A :

- (1) Connes' complex $CC_*^\lambda(A)$ which is defined as the cyclic coinvariants of the ordinary Hochschild complex.
- (2) The positive, negative and periodic cyclic bicomplexes $CC_*(A)$, $CN_*(A)$ and $CC_*^{\text{per}}(A)$.
- (3) The mixed complex $CM_*(A)$, which is also referred as “the (b, B) -complex” which also has two other variations: the negative mixed complex $CM_*^-(A)$ and the periodic mixed complex $CM_*^{\text{per}}(A)$.

As before, the cyclic bicomplexes and the mixed complexes are graded B -modules but are not necessarily differential graded B -modules. The obstruction to extending the graded B -module structure to a differential graded B -module structure stems from the fact that the cyclic permutations and the diagonal

B -action on the tensor powers of A do not necessarily commute. We will investigate the consequences of the operation of dividing out this obstruction on the cyclic complexes we mentioned above in a different paper [14] in a more general set-up where the complexes are twisted by some coefficient module. We would like to point out that the obstruction which prevents the cyclic bicomplex from being a differential graded B -module is a larger differential graded submodule in the sense that the “Hochschild subcomplex” or the “ b -subcomplex” of the Hopf cyclic bicomplex is a quotient of the Hopf–Hochschild complex we define here.

3. HOPF–HOCHSCHILD COHOMOLOGY

Definition 3.1. We define $A^e \rtimes B$ as $A^e \otimes B = A \otimes A^{op} \otimes B$ with the multiplication

$$(a_1 \otimes a'_1 \otimes b^1)(a_2 \otimes a'_2 \otimes b^2) = (ab_{(1)}^1(a_2) \otimes b_{(3)}^1(a'_2)a'_1 \otimes b_{(2)}^1b^2)$$

for any $(a_1 \otimes a'_1 \otimes b^1)$ and $(a_2 \otimes a'_2 \otimes b^2)$ from $A^e \rtimes B$.

Lemma 3.2. $A^e \rtimes B$ is a unital associative algebra.

Proof. For associativity, one must consider

$$\begin{aligned} & ((a_1 \otimes a'_1 \otimes b^1)(a_2 \otimes a'_2 \otimes b^2))(a_3 \otimes a'_3 \otimes b^3) \\ &= (a_1b_{(1)}^1(a_2) \otimes b_{(3)}^1(a'_2)a'_1 \otimes b_{(2)}^1b^2)(a_3 \otimes a'_3 \otimes b^3) \\ &= a_1b_{(1)}^1(a_2b_{(1)}^2(a_3)) \otimes b_{(3)}^1(b_{(3)}^2(a'_3)a'_2)a'_1 \otimes b_{(2)}^1b_{(2)}^2b^3 \\ &= (a_1 \otimes a'_1 \otimes b^1)((a_2 \otimes a'_2 \otimes b^2)(a_3 \otimes a'_3 \otimes b^3)) \end{aligned}$$

for any $(a_i \otimes a'_i \otimes b_i)$ from $A^e \rtimes B$ for $i = 1, 2, 3$. □

Lemma 3.3. V is a B -equivariant A -bimodule iff V is a $A^e \rtimes B$ -module.

Proof. V is a B -equivariant A -bimodule iff one has

$$\begin{aligned} (a_1 \otimes a_2 \otimes b)((a'_1 \otimes a'_2 \otimes b')(v)) &= a_1b(a'_1b'(v)a'_2)a_2 = a_1b_{(1)}(a'_1)(b_{(2)}b')(v)b_{(3)}(a'_2)a_2 \\ &= (a_1b_{(1)}(a'_1) \otimes b_{(3)}(a'_2)a_2) \cdot (b_{(2)}b')(v) \\ &= ((a_1 \otimes a_2 \otimes b)(a'_1 \otimes a'_2 \otimes b')) \cdot v \end{aligned}$$

for any $(a_1 \otimes a_2 \otimes b)$ and $(a'_1 \otimes a'_2 \otimes b')$ from $A^e \rtimes B$ and $v \in V$, i.e. V is a $A^e \rtimes B$ -module. □

Definition 3.4. Define a differential graded k -module $\text{CB}_*(A)$ by letting $\text{CB}_n(A) = A^{\otimes n+2}$ for any $n \geq 0$. Then we define pre-simplicial structure morphisms

$$\partial_j(a_0 \otimes \cdots \otimes a_{n+1}) = (a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1})$$

for any $n \geq 1$ and $0 \leq j \leq n$ and then define the differentials as $d_n^{\text{CB}} = \sum_{j=0}^n (-1)^j \partial_j$ for any $n \geq 1$.

Lemma 3.5. $\text{CB}_*(A)$ is a differential graded $A^e \rtimes B$ -module.

Proof. $\text{CB}_*(A)$ is a differential graded A -bimodule with the A -action defined as

$$a(a_0 \otimes \cdots \otimes a_{n+1}) = (aa_0 \otimes \cdots \otimes a_{n+1}) \quad (a_0 \otimes \cdots \otimes a_{n+1})a = (a_0 \otimes \cdots \otimes a_{n+1}a)$$

for any $a \in A$ and $(a_0 \otimes \cdots \otimes a_{n+1})$ from $\text{CB}_n(A)$. There is also a left B -module structure defined for $b \in B$ and $(a_0 \otimes \cdots \otimes a_{n+1})$ from $\text{CB}_n(A)$ as

$$b(a_0 \otimes \cdots \otimes a_{n+1}) = (b_{(1)}a_0 \otimes \cdots \otimes b_{(n+2)}a_{n+1})$$

This makes $\text{CB}_*(A)$ into a left $A^e \rtimes B$ -module since one has

$$\begin{aligned} b(a \otimes a')(a_0 \otimes \cdots \otimes a_{n+1}) &= (b_{(1)}(aa_0) \otimes b_{(2)}a_1 \otimes \cdots \otimes b_{(n+1)}a_n \otimes b_{(n+2)}(a_{n+1}a')) \\ &= (b_{(1)}(a)b_{(2)}(a_0) \otimes b_{(3)}a_1 \otimes \cdots \otimes b_{(n+3)}(a_{n+1})b_{(n+4)}(a')) \\ &= (b_{(1)}(a) \otimes b_{(3)}(a'))b_{(2)}(a_0 \otimes \cdots \otimes a_{n+1}) \end{aligned}$$

$b \in B$, $(a \otimes a')$ from A^e and $(a_0 \otimes \cdots \otimes a_{n+1})$ from $\text{CB}_n(A)$. □

Definition 3.6. Let A be a B -module algebra and let V be an B -equivariant A -bimodule. Define Hopf-Hochschild cochain complex of A with coefficients in V as

$$\text{CH}_{\text{Hopf}}^*(A, V) := \text{Hom}_{A^e \rtimes B}(\text{CB}_*(A), V)$$

$HH_{\text{Hopf}}^*(A, V)$ Hopf-Hochschild cohomology of A with coefficients in V is defined to be the cohomology of this cochain complex.

Theorem 3.7. Let A be an H -unital projective B -module algebra and assume V is an arbitrary B -equivariant A -bimodule. Then one has isomorphisms of the form

$$HH_{\text{Hopf}}^{n+1}(A, V) \cong \text{Ext}_{A^e \rtimes B}^n(\Omega(A), V)$$

for any $n \geq 1$.

Proof. We have a short exact sequence of $A^e \rtimes B$ -modules of the form

$$0 \rightarrow \Omega(A) \rightarrow A^e \xrightarrow{\mu_A} A \rightarrow 0$$

where μ_A denotes the multiplication morphism. Since A is B -projective, the brutal truncation $\text{CB}_{*>0}(A)$ of $\text{CB}_*(A)$ at $n = 0$ is a $A^e \rtimes B$ -projective resolution of $\Omega(A)$ the kernel of the multiplication map. The result follows. □

Definition 3.8. Assume A is a B -module algebra and V is a B -equivariant A -module. A morphism of k -modules $A \xrightarrow{D} V$ is called a V -valued derivation on A iff

$$D(aa') = D(a)a' + aD(a')$$

for any $a, a' \in A$. The same derivation is called a V -valued B -equivariant derivation if

$$D(ba) = bD(a)$$

for any $a \in A$ and $b \in B$. The k -module of V -valued and V -valued B -equivariant derivations on A are denoted by $\text{Der}(A, V)$ and $\text{Der}_B(A, V)$ respectively.

Let $v \in V$ be fixed and consider the k -module morphism $A \xrightarrow{[v, \cdot]} V$ defined by $[v, a] = va - av$ for any $a \in A$. Then

$$[v, aa'] = vaa' - aa'v = vaa' - av a' + av a' - aa'v = [v, a]a' + a[v, a']$$

for any $a, a' \in A$ and $b \in B$. This means elements of the form $[v, \cdot]$ from $\text{Hom}_k(A, V)$ are V -valued derivations on A . However if we were to require $[v, \cdot]$ to be a B -equivariant derivation, then we need to have

$$[v, b(a)] = vb(a) - b(a)v = b_{(1)}(v)b_{(2)}(a) - b_{(1)}(a)b_{(2)}(v) = b[v, a]$$

for any $a \in A, b \in B$. In case V is a trivial B -module via the counit, i.e. $b(v) = \varepsilon(b)v$ then the condition above is satisfied.

Proposition 3.9. Let A be an B -module algebra and V be an B -equivariant A -module as before. Then one has

$$HH_{\text{Hopf}}^0(A, V) \cong ({}^B V)^{\text{Lie}(A)} \quad \text{and} \quad HH_{\text{Hopf}}^1(A, V) \cong \text{Der}_B(A, V)/[A, {}^B V]$$

Proof. If we consider $f \in \text{Hom}_{A^e \rtimes B}(\text{CB}_n(A), V)$ we see that

$$f(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) a_{n+1}$$

Since $b(1_A) = \varepsilon(b)$ for all $b \in B$, we see that $\text{CH}_{\text{Hopf}}^0(A, V) \cong \text{Hom}_B(k, V) \cong {}^B V$ where ${}^B V$ is defined as the submodule $\{v \in V \mid b(v) = \varepsilon(b)v, b \in B\}$ of V . The proof that $\text{CH}_*^1(A, V) \cong \text{Hom}_B(A, V)$ is similar. Then $v \in HH_{\text{Hopf}}^0(A, V)$ iff

$$(3.1) \quad d_{\text{CH}}^0(v)(1 \otimes a \otimes 1) = v(a \otimes 1) - v(1 \otimes a) = av - va = 0$$

i.e. $[a, v] = 0$ for any $a \in A$ which is the same as the invariants of the adjoint action of $\text{Lie}(A)$ on ${}^B V$, i.e. $({}^B V)^{\text{Lie}(A)}$. Similarly, $f \in \ker(d_{\text{CH}}^1)$ iff

$$\begin{aligned} (d_{\text{CH}}^1 f)(1 \otimes a \otimes a' \otimes 1) &= f(a \otimes a' \otimes 1) - f(1 \otimes aa' \otimes 1) + f(1 \otimes a \otimes a') \\ &= af(1 \otimes a' \otimes 1) - f(1 \otimes aa' \otimes 1) + f(1 \otimes a \otimes 1)a' = 0 \end{aligned}$$

for any $(1 \otimes a \otimes a' \otimes 1)$ from $\text{CB}_2(A)$. In other words $f \in \ker(d_{\text{CH}}^1)$ iff $D_f(a) := f(1 \otimes a \otimes 1)$ is a derivation. Moreover, since f is B -equivariant, so is D_f . Equation 3.1 tells us that the image of d_{CH}^0 consists of the elements of the form $[a, v]$ where $v \in {}^B V$ and $a \in A$. In other words $HH_{\text{Hopf}}^1(A, V) \cong \text{Der}_B(A, V)/[A, {}^B V]$. \square

4. HOPF-HOCHSCHILD HOMOLOGY REVISITED

Definition 4.1. Assume B is a Hopf algebra and let U be a right B -module. Then one can think of U as a left module via the action $b \cdot u := u \cdot S(b)$ for any $u \in U$ and $b \in B$. We denote the new module by U^{op} .

Theorem 4.2. Assume B is a Hopf algebra with an invertible antipode. Let A be a B -module algebra and V be a B -equivariant A -bimodule. Then ${}_B \text{QCH}_*(A, B, V)$ is isomorphic to $V^{op} \otimes_{A^e \rtimes B} \text{CB}_*(A)$ as differential graded k -modules.

Proof. Define a morphism of graded modules

$$\text{CH}_*(A, V) \xrightarrow{\varphi_*} V^{op} \otimes_{A^e \rtimes B} \text{CB}_*(A)$$

by letting

$$\varphi_n(a_1 \otimes \cdots \otimes a_n \otimes v) = v \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$$

for any $(a_1 \otimes \cdots \otimes a_n \otimes v)$ from $\text{CH}_n(A, V)$. Notice that

$$\begin{aligned} \varphi L_b(a_1 \otimes \cdots \otimes a_n \otimes v) &= v S^{-1}(b_{(2)}) \otimes_{A^e \rtimes B} (1 \otimes L_{b_{(1)}}(a_1 \otimes \cdots \otimes a_n) \otimes 1) \\ &= v S^{-1}(b_{(2)}) \otimes_{A^e \rtimes B} L_{b_{(1)}}(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \\ &= \varepsilon(b) v \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \end{aligned}$$

by observing $x(1_A) = \varepsilon(x)$ for any $x \in B$. Therefore φ_* factors as

$$\text{CH}_*(A, V) \xrightarrow{q'_*} {}_B \text{CH}_*(A, V) \xrightarrow{\varphi'_*} V^{op} \otimes_{A^e \rtimes B} \text{CB}_*(A)$$

where q'_* is the canonical quotient taking $\text{CH}_*(A, V)$ to the graded module of B -coinvariants ${}_B \text{CH}_*(A, V)$.

However, for any $b \in B$ we also have

$$\varphi_n[L_b, \partial_{n+1}] = \varphi_n L_b \partial_{n+1} - \varphi_n \partial_{n+1} L_b = \varphi_n \partial_{n+1} (\varepsilon(b) - L_b)$$

Together with the fact that $\varphi_n \partial_{n+1} = (id_V \otimes \partial_0) \varphi_{n+1}$ we see

$$\varphi_n[L_b, \partial_{n+1}] = (id_V \otimes \partial_0) \varphi_{n+1} (\varepsilon(b) - L_b) = 0$$

for any $b \in B$. This implies we get an extension

$$\begin{array}{ccccc} \mathrm{CH}_*(A, V) & \xrightarrow{q'_*} & {}_B\mathrm{CH}_*(A, V) & \xrightarrow{\varphi'_*} & V^{op} \otimes_{A^e \rtimes B} \mathrm{CB}_*(A) \\ q_* \downarrow & & {}_Bq_* \downarrow \cong & & \parallel \\ \mathrm{QCH}_*(A, B, V) & \xrightarrow{q'_*} & {}_B\mathrm{QCH}_*(A, B, V) & \xrightarrow{\varphi''_*} & V^{op} \otimes_{A^e \rtimes B} \mathrm{CB}_*(A) \end{array}$$

Since φ_* is a morphism of differential graded k -modules so is φ''_* because both q_* and q'_* are morphisms of differential graded k -modules too. Notice that ${}_Bq_*$ is an isomorphism of graded k -modules since q'_* factors through $\mathrm{QCH}_*(A, B, V)$. Also, observe that $V^{op} \otimes_{A^e \rtimes B} \mathrm{CB}_n(A)$ is generated by elements of the form $(v \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1))$ since

$$\begin{aligned} & (v \otimes_{A^e \rtimes B} (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1})) \\ &= (v \otimes_{A^e \rtimes B} (a_0 \otimes a_{n+1})(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)) \\ &= (a_{n+1}va_0 \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)) \end{aligned}$$

for some $v \in V$ and $a_i \in A$. This implies φ''_* is an epimorphism. Now define a section

$$V^{op} \otimes_{A^e \rtimes B} \mathrm{CB}_*(A) \xrightarrow{s_*} {}_B\mathrm{QCH}_*(A, B, V)$$

by letting

$$s_n(v \otimes_{A^e \rtimes B} (a \otimes a_1 \otimes \cdots \otimes a_n \otimes a')) = [a_1 \otimes \cdots \otimes a_n \otimes a'va]$$

for any $(v \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1))$ from $V^{op} \otimes_{A^e \rtimes B} \mathrm{CB}_*(A)$. The section s_* is well-defined since

$$\begin{aligned} & s_n \left(v(a \otimes a' \otimes b) \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \right) \\ &= s_n \left(S(b)(a'va) \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \right) \\ &= [a_1 \otimes \cdots \otimes a_n \otimes S(b)(a'va)] = [S(b_{(2)})b_{(3)}(a_1 \otimes \cdots \otimes a_n) \otimes S(b_{(1)})(a'va)] \\ &= [b(a_1 \otimes \cdots \otimes a_n) \otimes a'va] = s_n \left(v \otimes_{A^e \rtimes B} (a \otimes b(a_1 \otimes \cdots \otimes a_n) \otimes a') \right) \\ &= s_n \left(v \otimes_{A^e \rtimes B} (a \otimes a' \otimes b)(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \right) \end{aligned}$$

for any $(v \otimes_{A^e \rtimes B} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1))$ from $V^{op} \otimes_{A^e \rtimes B} \mathrm{CB}_*(A)$ and $(a \otimes a' \otimes b)$ from $A^e \rtimes B$. We used the notation $[\Psi]$ to denote the class of an element $\Psi \in \mathrm{QCH}_*(A, V)$ in ${}_B\mathrm{QCH}_*(A, V)$. One can easily see that $s_*\varphi'_* = id_*$ which means φ''_* is also a monomorphism. The result follows. \square

Corollary 4.3. *Let B be a Hopf algebra with an invertible antipode. Assume A is a B -module algebra and V is a left B -equivariant A -bimodule. Then*

$$HH_0^{\mathrm{Hopf}}(A, V^{op}) \cong {}_B V/[A, {}_B V]$$

where ${}_B V := k \otimes_B V \cong A^e \otimes_{A^e \rtimes B} V$.

Corollary 4.4. *Let B be a Hopf algebra with an invertible antipode. Assume A is an H -unital projective B -module algebra and V is a B -equivariant A -bimodule. Then one has isomorphisms of the form*

$$HH_{n+1}^{\text{Hopf}}(A, V) \cong \text{Tor}_n^{A^e \rtimes B}(\Omega(A), V^{op})$$

for any $n \geq 1$.

Proof. Consider the short exact sequence of $A^e \rtimes B$ -modules

$$0 \rightarrow \Omega(A) \rightarrow A^e \xrightarrow{\mu_A} A \rightarrow 0$$

where μ_A is the multiplication on A . If A is H -unital and A is B -projective then the brutal truncation $\text{CB}_{*>0}(A)$ of $\text{CB}_*(A)$ at $n = 0$ is a $A^e \rtimes B$ -projective resolution of $\Omega(A)$. The rest of the proof is similar to that of Theorem 3.7. \square

Remark 4.5. After Theorem 4.2 and Corollary 4.4, one can see that there is another possible definition of the Hopf-Hochschild complex of a B -module algebra A . Namely, one can use the differential graded k -module

$$V \otimes_{A^e \rtimes B} \text{CB}_*(A)$$

to define the Hopf-Hochschild homology of A for a right B -equivariant A -bimodule V .

5. CATEGORICAL ALGEBRA AND COFINALITY

Definition 5.1. A small category \mathcal{C} is called k -linear if for each $X, Y \in \text{Ob}(\mathcal{C})$, the Hom object $\text{Hom}_{\mathcal{C}}(X, Y)$ is a k -module and the composition maps

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$

are k -bilinear for any $X, Y, Z \in \text{Ob}(\mathcal{C})$. A functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ between two k -linear categories is called a k -linear functor if the map

$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F_{X,Y}} \text{Hom}_{\mathcal{C}'}(X, Y)$$

is a morphism of k -modules for any X, Y in $\text{Ob}(\mathcal{C})$.

Definition 5.2. A k -linear bifunctor on \mathcal{C} with values in another k -linear category \mathcal{C}' is just a functor of the form $\mathcal{C} \times \mathcal{C}^{op} \xrightarrow{\mathcal{H}} \mathcal{C}'$.

Definition 5.3. Let \mathcal{C} be a k -linear small category and let \mathcal{H} be a bifunctor on \mathcal{C} with values in $k\text{-}\mathbf{Mod}$. Define $\text{CH}_*(\mathcal{C}, \mathcal{H})$ the Hochschild complex of \mathcal{C} with coefficients in the bifunctor \mathcal{H} as the differential graded k -module given by

$$\text{CH}_n(\mathcal{C}, \mathcal{H}) := \bigoplus_{X_0, \dots, X_n} \mathcal{H}(X_0, X_n) \otimes \text{Hom}_{\mathcal{C}}(X_1, X_0) \otimes \dots \otimes \text{Hom}_{\mathcal{C}}(X_n, X_{n-1})$$

with a pre-simplicial structure

$$\begin{aligned} \partial_j(h \otimes X_0 \xleftarrow{u_1} \dots \xleftarrow{u_n} X_n) \\ = \begin{cases} \mathcal{H}(u_1, id_{X_n})(h) \otimes X_1 \xleftarrow{u_2} \dots \xleftarrow{u_n} X_n & \text{if } j = 0 \\ h \otimes \dots \xleftarrow{u_{i-1}} X_{i-1} \xleftarrow{u_i u_{i+1}} X_{i+1} \xleftarrow{u_{i+2}} \dots & \text{if } 0 < j < n-1 \\ \mathcal{H}(id_{X_0}, u_n)(h) \otimes X_0 \xleftarrow{u_1} \dots \xleftarrow{u_{n-1}} X_{n-1} & \text{if } j = n \end{cases} \end{aligned}$$

defined for any $n \geq 1$ and $(h \otimes X_0 \xleftarrow{u_1} \dots \xleftarrow{u_n} X_n)$ from $\text{CH}_n(\mathcal{C}, \mathcal{H})$. In the case $\mathcal{H} = \text{Hom}_{\mathcal{C}}$, we denote $\text{CH}_*(\mathcal{C}, \text{Hom}_{\mathcal{C}})$ simply by $\text{CH}_*(\mathcal{C})$.

Assume \mathcal{A}_* and \mathcal{B}_* are two pre-simplicial k -modules and let $f, g : \mathcal{A}_* \rightarrow \mathcal{B}_*$ be two morphisms of pre-simplicial modules. Now, recall from [15] that a pre-simplicial homotopy h_* between f_* and g_* is a set of k -module morphisms $h_i : \mathcal{A}_n \rightarrow \mathcal{B}_{n+1}$ defined for $0 \leq i \leq n$ satisfying

$$h_i \partial_j = \partial_j h_{i+1}, \quad \text{if } j \leq i \quad \quad h_i \partial_j = \partial_{j+1} h_i, \quad \text{if } j \geq i+1 \quad \quad \partial_i h_i = \partial_i h_{i-1}$$

where $f_* = \partial_0 h_0$ and $g_* = \partial_{*+1} h_*$.

Definition 5.4. Let \mathcal{C} be a k -linear category and let \mathcal{D} be a k -linear subcategory of \mathcal{C} . Then \mathcal{D} is called a cofinal subcategory of \mathcal{C} if for every object C of \mathcal{C} , there exists an object D in \mathcal{D} and a retract $D \xrightarrow{r} C$ in \mathcal{C} .

Theorem 5.5. Let \mathcal{C} be a small k -linear category and \mathcal{H} be a bifunctor on \mathcal{C} with values in $k\text{-}\mathbf{Mod}$. Assume \mathcal{D} is a cofinal subcategory of \mathcal{C} . Then the natural inclusion $\text{CH}_*(\mathcal{D}, \mathcal{H}) \xrightarrow{i_*} \text{CH}_*(\mathcal{C}, \mathcal{H})$ is a homotopy equivalence.

Proof. For every object C in \mathcal{C} fix a choice of object $\delta(C)$ and a retract $\delta(C) \xrightarrow{r(C)} C$ such that for each object D in \mathcal{D} , the choice is $D \xrightarrow{id_D} D$. Denote the left inverse of $r(C)$ by $s(C)$ for every C in \mathcal{C} . Now define a morphism of differential graded k -modules $\text{CH}_*(\mathcal{C}, \mathcal{H}) \xrightarrow{M_*} \text{CH}_*(\mathcal{D}, \mathcal{H})$ by letting

$$M_*(h \otimes X_0 \xleftarrow{u_1} \dots \xleftarrow{u_n} X_n) = \mathcal{H}(s_n, r_0)(h) \otimes \delta(X_0) \xleftarrow{s_0 u_1 r_1} \dots \xleftarrow{s_{n-1} u_n r_n} \delta(X_n)$$

for any $(h \otimes X_0 \xleftarrow{u_1} \dots \xleftarrow{u_n} X_n)$ in $\text{CH}_n(\mathcal{C}, \mathcal{H})$ where we use $r_i = r(X_i)$ and $s_i = s(X_i)$ for any $0 \leq i \leq n$. It is easy to see that M_* is a morphism of pre-simplicial modules since $r_i s_i = id_i$. Note that

the composition M_*i_* is identity on $\mathrm{CH}_*(\mathcal{D}, \mathcal{H})$. We are going to show i_*M_* is homotopic to the identity on $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$. We define a pre-simplicial homotopy by letting

$$\begin{aligned} h_i(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n) \\ = \mathcal{H}(s_n, id_{X_0})(h) \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_i} X_i \xleftarrow{r_i} \delta(X_i) \xleftarrow{s_i u_{i+1} r_{i+1}} \cdots \xleftarrow{s_{n-1} u_n r_n} \delta(X_n) \end{aligned}$$

for any $0 \leq i \leq n$ and for any $(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n)$ in $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$. Note that $\partial_0 h_0 = i_*M_*$ and $\partial_{n+1} h_n = id_*$. We leave the verification of pre-simplicial homotopy identities to the reader. \square

Corollary 5.6. *If $\mathcal{C}' \xrightarrow{F} \mathcal{C}$ is a k -linear functor, then one has a morphism of differential graded k -modules of the form*

$$\mathrm{CH}_*(\mathcal{C}', \mathcal{H}F) \xrightarrow{F_*} \mathrm{CH}_*(\mathcal{C}, \mathcal{H})$$

for any bifunctor \mathcal{H} on \mathcal{C} with values in $k\text{-}\mathbf{Mod}$. Moreover, if F is an equivalence of categories then F_* is a homotopy equivalence.

Proof. First, let us explain what $\mathrm{CH}_*(\mathcal{C}', \mathcal{H}F)$ is: $\mathrm{CH}_n(\mathcal{C}', \mathcal{H}F)$ generated by homogeneous tensors of the form $(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n)$ where $h \in \mathcal{H}(F(X_n), F(X_0))$ for any $n \geq 0$. The “action” of u_1 and u_n on h are defined through F . The morphism F_* of pre-simplicial k -modules is defined as

$$F_n(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n) = h \otimes F(X_0) \xleftarrow{F(u_1)} \cdots \xleftarrow{F(u_n)} F(X_n)$$

for any $n \geq 0$ and $(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n)$ from $\mathrm{CH}_n(\mathcal{C}', \mathcal{H}F)$. Now assume F is an equivalence with a quasi-inverse G and with the isomorphism $id_{\mathcal{C}'} \xrightarrow{\varphi} GF$. Note that we have the composition

$$\mathrm{CH}_*(\mathcal{C}', \mathcal{H}FGF) \xrightarrow{G_*F_*} \mathrm{CH}_*(\mathcal{C}', \mathcal{H}F)$$

and the image of the functor $\mathcal{C}' \xrightarrow{GF} \mathcal{C}'$ is a cofinal subcategory of \mathcal{C}' since $GF \simeq id_{\mathcal{C}'}$. Thus $G_*F_* \simeq id_*$. The same argument works also for $\mathcal{C} \xrightarrow{FG} \mathcal{C}$ and we see that $F_*G_* \simeq id_*$. The result follows. \square

Definition 5.7. Let \mathcal{C} be a k -linear category which has finite coproducts and let \mathcal{D} and \mathcal{E} be two full k -linear subcategories. \mathcal{D} is said to generate \mathcal{E} if for every object E of \mathcal{E} there is a natural number $n \geq 1$ and a set of objects D_1, \dots, D_n of \mathcal{D} such that $E \cong \coprod_{i=1}^n D_i$.

Theorem 5.8. *Let \mathcal{C} , \mathcal{D} and \mathcal{E} be as in Definition 5.7. Then the natural inclusion $\mathrm{CH}_*(\mathcal{D}, \mathcal{H}) \xrightarrow{i_*} \mathrm{CH}_*(\mathcal{E}, \mathcal{H})$ is a homotopy equivalence for any bifunctor \mathcal{H} on \mathcal{C} with values in $k\text{-}\mathbf{Mod}$.*

Proof. Take an object E from \mathcal{E} and consider “ \mathcal{D} -components” $\{D_1, \dots, D_n\}$ of E . Since $E \cong \coprod_{i=1}^n D_i$ and

$$\mathrm{Hom}_{\mathcal{C}}(E, E) \cong \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(D_i, D_j)$$

there are morphisms $E \xleftarrow{v_i} D_i$ and $D_j \xleftarrow{u_j} E$ such that $\sum_i v_i u_i = id_E$. Now take $h \otimes E_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} E_n$ from $\mathrm{CH}_n(\mathcal{E}, \mathcal{H})$ and let D_j^i be the \mathcal{D} -components of E_i , and $E_i \xleftarrow{v_j^i} D_j^i \xleftarrow{u_j^i} E_i$ be the corresponding

splitting of id_{E_i} for $0 \leq i \leq n$. Define a morphism of pre-simplicial modules $CH_*(\mathcal{E}, \mathcal{H}) \xrightarrow{M_*} CH_*(\mathcal{D}, \mathcal{H})$ by letting

$$\begin{aligned} M_n(h \otimes E_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} E_n) \\ = \sum_{i_0, \dots, i_n} \mathcal{H}(u_{i_n}^n, v_{i_0}^0)(h) \otimes D_{i_0}^0 \xleftarrow{u_{i_0}^0 f_1 v_{i_1}^1} \cdots \xleftarrow{u_{i_{n-1}}^{n-1} f_n v_{i_n}^n} D_{i_n}^n \end{aligned}$$

Notice that $M_* i_*$ is the identity on $CH_*(\mathcal{D}, \mathcal{H})$. Observe also that the identity $\sum_j v_j^i u_j^i = id_{E_i}$ implies M_* is a morphism of pre-simplicial modules. Although $i_* M_*$ is not identity, we will furnish a pre-simplicial homotopy between id_* and $i_* M_*$ on $CH_*(\mathcal{E}, \mathcal{H})$. We let

$$\begin{aligned} h_s(h \otimes E_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} E_n) \\ = \mathcal{H}(id_{E_n}, v_{i_0}^0)(h) \otimes D_{i_0}^0 \xleftarrow{u_{i_0}^0 f_1 v_{i_1}^1} \cdots \xleftarrow{u_{i_{s-1}}^{s-1} f_s v_{i_s}^s} D_{i_s}^s \xleftarrow{u_{i_s}^s} E_s \xleftarrow{f_{s+1}} \cdots \xleftarrow{f_n} E_n \end{aligned}$$

for $n \geq 0$, $0 \leq s \leq n$ and $h \otimes E_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} E_n$ from $CH_n(\mathcal{E}, \mathcal{H})$. We leave the verification of the pre-simplicial homotopy identities to the reader. \square

6. B -CATEGORIES AND EQUIVARIANT BIFUNCTORS

Definition 6.1. A k -linear category \mathcal{C} is called a B -category if each $\text{Hom}_{\mathcal{C}}(X, Y)$ is a left B -module and the composition

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$

is a B -module morphism via the diagonal action of B , for any X, Y, Z taken from $Ob(\mathcal{C})$. In other words, one has $b(gf) = b_{(1)}(g)b_{(2)}(f)$ for any $b \in B$, $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$. A functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ between two B -categories is called B -equivariant if the structure morphisms

$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F_{X,Y}} \text{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

are B -module morphisms for any $X, Y \in Ob(\mathcal{C})$.

Remark 6.2. It came to our attention that Cibils and Solotar defined the same notion in [6] but they called the same object as B -module category. Their primary example of the underlying bialgebra is a group ring which is cocommutative. The bialgebras, or in general Hopf algebras, we consider are not necessarily cocommutative.

Example 6.3. Assume that B is a Hopf algebra and A is a B -module algebra. Consider the category \mathbf{mod}_{B-A} of left B -equivariant right A -modules and all A -linear morphisms. Note that we do consider **all** A -module morphisms, not just B -equivariant A -module morphisms. Define a left B action of $\text{Hom}_{\mathbf{mod}_{B-A}}(X, Y)$ by letting

$$(bf)(x) = b_{(1)}f(S(b_{(2)})x)$$

for any $f \in \text{Hom}_{\mathbf{mod}_{B-A}}(X, Y)$ and $x \in X$. However, one needs to show that bf is still a right A -module morphism for any $f \in \text{Hom}_{\mathbf{mod}_{B-A}}(X, Y)$ and $b \in B$. Therefore we check

$$\begin{aligned} (bf)(xa) &= b_{(1)}f(S(b_{(2)})(xa)) = b_{(1)}f(S(b_{(3)})(x)S(b_{(2)})(a)) \\ &= b_{(1)}f(S(b_{(4)})(x))b_{(2)}S(b_{(3)})(a) = (bf)(x)a \end{aligned}$$

for any $a \in A$, $b \in B$, $x \in X$ and $f \in \text{Hom}_{\mathbf{mod}_{B-A}}(X, Y)$. Now notice that for any $g \in \text{Hom}_{\mathbf{mod}_{B-A}}(Y, Z)$ one has

$$b(gf)(x) = b_{(1)}gf(S(b_{(2)})(x)) = b_{(1)}g(S(b_{(2)})b_{(3)}f(S(b_{(4)})(x))) = (b_{(1)}g)(b_{(2)}f)(x)$$

for any $x \in X$. In other words \mathbf{mod}_{B-A} is a B -category.

Example 6.4. Assume B is a Hopf algebra and A is a B -module algebra. Let \mathbf{proj}_B-A be the full subcategory of \mathbf{mod}_{B-A} consisting of finitely generated left B -equivariant projective right A -modules. Then \mathbf{proj}_B-A is a B -category.

Example 6.5. Assume B is a Hopf algebra and A is a B -module algebra. Consider the full subcategory $*_B^A$ of \mathbf{free}_B-A which consists of one single object A considered as a right A -module via the right regular representation. Then $*_B^A$ is a B -category.

Definition 6.6. A bifunctor \mathcal{H} on a B -category \mathcal{C} with values in $B\text{-Mod}$ is called a B -equivariant bifunctor if the structure morphisms

$$(6.1) \quad \mathcal{H}(X, Y) \otimes \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \mathcal{H}(X, Z) \quad \text{Hom}_{\mathcal{C}}(W, X) \otimes \mathcal{H}(X, Y) \rightarrow \mathcal{H}(W, Y)$$

are B -module morphisms where B acts diagonally on the left. In other words

$$b(\mathcal{H}(u, v)(h)) = \mathcal{H}(b_{(1)}(u), b_{(3)}(v))(b_{(2)}(h))$$

for any $u \in \text{Hom}_{\mathcal{C}}(W, X)$, $v \in \text{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathcal{H}(X, Y)$, and $b \in B$.

Example 6.7. For a B -category \mathcal{C} , the bifunctor $\mathcal{H} = \text{Hom}_{\mathcal{C}}(\cdot, \cdot)$ is a B -equivariant bifunctor on \mathcal{C} with values in $B\text{-Mod}$.

Definition 6.8. For a B -category \mathcal{C} , let ${}^B\mathcal{C}$ denote the subcategory of morphisms of \mathcal{C} which are B -invariant, i.e. $X \xrightarrow{f} Y$ belongs to ${}^B\mathcal{C}$ iff $b(f) = \varepsilon(b)f$ for any $b \in B$.

Remark 6.9. Assume B is a Hopf algebra and \mathbf{mod}_{B-A} be the B -category defined in Example 6.3. Then one can see that ${}^B\mathbf{mod}_{B-A}$, the subcategory of B -invariant morphisms, is the category of left B -equivariant right A -modules and their B -equivariant A -module morphisms since $bf = \varepsilon(b)f$ iff f is B -equivariant. We would like to note that for the B -equivariant homotopical invariants of \mathbf{mod}_{B-A} the subcategory ${}^B\mathbf{mod}_{B-A}$ is rather small. The situation is very similar to topological spaces admitting an action of a fixed group G . The G -equivariant homotopical invariants of a G -space X are computed via $\mathcal{B}(G, X) := EG \wedge_G X$ rather than $X/G \simeq * \wedge_G X$. Similarly, ${}^B\mathbf{mod}_{B-A}$ should be considered as

the lowest order equivariant invariant of $\mathbf{mod}_B\text{-}A$. Thus we propose that for higher order equivariant homotopical invariants of a B -module algebra A , such as equivariant K -theoretical, Hochschild and cyclic homological invariants, one should use the B -category $\mathbf{mod}_B\text{-}A$ of B -equivariant modules and their A -linear morphisms, or its various B -subcategories, instead of using simply ${}^B\mathbf{mod}_B\text{-}A$ the subcategory of B -equivariant A -module morphisms of $\mathbf{mod}_B\text{-}A$. Our justification lies in Section 7 where we prove Morita invariance in Corollary 7.9.

7. MORITA INVARIANCE

Lemma 7.1. *If \mathcal{C} is a B -category and \mathcal{H} is a bifunctor on \mathcal{C} with values in $B\text{-}\mathbf{Mod}$, then $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$ is a graded B -module. However, $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$ is not a pre-simplicial B -module unless B is cocommutative.*

Proof. The B -action of $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$ is defined diagonally, i.e.

$$L_b(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n) := b_{(1)}(h) \otimes X_0 \xleftarrow{b_{(2)}u_1} \cdots \xleftarrow{b_{(n+1)}u_n} X_n$$

for any $b \in B$, $n \geq 0$ and $h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n$ from $\mathrm{CH}_n(\mathcal{C}, \mathcal{H})$. The fact that $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$ is NOT a pre-simplicial B -module is because of the last face morphism: one has

$$\begin{aligned} \partial_n L_b(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n) \\ = \mathcal{H}(b_{(n+1)}(f_n), id_{X_0})(b_{(1)}h) \otimes X_0 \xleftarrow{b_{(2)}u_1} \cdots \xleftarrow{b_{(n)}u_{n-1}} X_{n-1} \end{aligned}$$

which is different than

$$\begin{aligned} L_b \partial_n(h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n) \\ = \mathcal{H}(b_{(1)}(f_n), id_{X_0})(b_{(2)}h) \otimes X_0 \xleftarrow{b_{(3)}u_1} \cdots \xleftarrow{b_{(n+1)}u_{n-1}} X_{n-1} \end{aligned}$$

for any $n \geq 0$, $b \in B$ and $h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n$ from $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$, unless B is cocommutative. \square

Definition 7.2. Define a graded k -submodule $J_*(\mathcal{C}, B, \mathcal{H})$ of $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$ generated by elements of the form

$$[L_b, \partial_n](h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n)$$

where $n \geq 1$, $b \in B$ and $h \otimes X_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} X_n$ from $\mathrm{CH}_n(\mathcal{C}, \mathcal{H})$.

Lemma 7.3. *$J_*(\mathcal{C}, B, \mathcal{H})$ is a differential graded k -submodule and graded B -submodule of $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})$. Therefore $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})/J_*(\mathcal{C}, B, \mathcal{H})$ is a differential graded B -module.*

Proof. The proof is identical to that of Proposition 2.10. \square

Definition 7.4. Let $\mathrm{QCH}_*(\mathcal{C}, B, \mathcal{H})$ be the quotient $\mathrm{CH}_*(\mathcal{C}, \mathcal{H})/J_*(\mathcal{C}, B, \mathcal{H})$.

Theorem 7.5. *Let \mathcal{D} be a cofinal B -subcategory of \mathcal{C} . Then for any B -equivariant bifunctor \mathcal{H} on \mathcal{C} with values in $B\text{-Mod}$ then the natural inclusion*

$$\mathrm{QCH}_*(\mathcal{D}, B, \mathcal{H}) \xrightarrow{i_*} \mathrm{QCH}_*(\mathcal{C}, B, \mathcal{H})$$

is a homotopy equivalence.

Proof. The proof is almost verbatim that of Theorem 5.5 after noticing $J_*(\mathcal{C}, B, \mathcal{H})$ is stable under the pre-simplicial homotopy we furnished there. \square

Corollary 7.6. *Let \mathcal{C} and \mathcal{C}' be two B -categories and let $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ be a functor of B -categories. Then for any B -equivariant bifunctor \mathcal{H} on \mathcal{C}' one has a morphism of differential graded B -modules of the form*

$$\mathrm{QCH}_*(\mathcal{C}, B, \mathcal{H}F) \xrightarrow{F_*} \mathrm{QCH}_*(\mathcal{C}', B, \mathcal{H})$$

Moreover, if F is an equivalence of B -categories, then F_ is a homotopy equivalence.*

Corollary 7.7. *Assume A is a B -module algebra. Then $\mathbf{free}_B A$ is a cofinal B -subcategory of $\mathbf{proj}_B A$. Furthermore, the natural inclusion functor $\mathbf{free}_B A \rightarrow \mathbf{proj}_B A$ induces a homotopy equivalence of differential graded B -modules of the form*

$$\mathrm{QCH}_*(\mathbf{free}_B A, B, \mathcal{H}) \xrightarrow{i_*} \mathrm{QCH}_*(\mathbf{proj}_B A, B, \mathcal{H})$$

for any B -equivariant bifunctor \mathcal{H} on $\mathbf{proj}_B A$ with values in $B\text{-Mod}$.

Corollary 7.8. *Assume A is a B -module algebra. The subcategory $*_B^A$ freely generates the B -subcategory $\mathbf{free}_B A$ of $\mathbf{proj}_B A$. Then the natural inclusion*

$$\mathrm{QCH}_*(*_B^A, B, \mathcal{H}) \xrightarrow{i_*} \mathrm{QCH}_*(\mathbf{free}_B A, B, \mathcal{H})$$

is a homotopy equivalence of differential graded B -modules for any B -equivariant bifunctor \mathcal{H} defined on $\mathbf{proj}_B A$ with values in $B\text{-Mod}$. Furthermore, $\mathrm{QCH}_(*_B^A, B, \mathcal{H})$ and $\mathrm{QCH}_*(A, B, \mathcal{H}(A, A))$ are isomorphic as the differential graded B -modules.*

Proof. The proof relies on the fact that $\mathrm{Hom}_{\mathbf{proj}_B A}(A, A) \cong A$. The rest of the proof is trivial. \square

Theorem 7.9 (Morita invariance for Hopf-Hochschild (co)homology). *Let B be a Hopf algebra and assume that A and A' are two B -module algebras. If $\mathbf{mod}_B A$ and $\mathbf{mod}_B A'$ the category of finitely generated B -equivariant representations of A and A' respectively are B -equivariantly equivalent, then Hopf-Hochschild complex ${}_B\mathrm{QCH}_*(A, B, A)$ of A and Hopf-Hochschild complex ${}_B\mathrm{QCH}_*(A', B, A')$ of A' are quasi-isomorphic.*

8. TWISTED EQUIVARIANT BIFUNCTORS AS COEFFICIENTS

In this section we assume B is a Hopf algebra with an invertible antipode.

Definition 8.1. An arbitrary left-left B -module/comodule M is called a Yetter Drinfeld module if one has

$$(bm)_{(-1)} \otimes (bm)_{(0)} = b_{(1)}m_{(-1)}S(b_{(3)}) \otimes b_{(2)}m_{(0)}$$

for any $b \in B$ and $m \in M$ where we use Sweedler's notation for the coproduct on H and for the H -coaction on M .

Definition 8.2. Assume M is a Yetter-Drinfeld module over B . Let \mathcal{C} be an arbitrary B -category and \mathcal{H} be an arbitrary B -equivariant bifunctor on \mathcal{C} with values in $B\text{-}\mathbf{Mod}$. We define a new bifunctor $M \ltimes \mathcal{H}$ by letting $M \ltimes \mathcal{H}(X, Y) := M \otimes \mathcal{H}(X, Y)$ on the objects for any $X, Y \in \text{Ob}(\mathcal{C})$. Now we let

$$b(m \otimes h) = b_{(1)}(m) \otimes b_{(2)}(h)$$

for any $m \otimes h$ in $\mathcal{H}(X, Y)$. We define the bifunctor $M \ltimes \mathcal{H}$ on morphisms as follows: notice that such functors are defined by structure morphisms given in (6.1). Then we let

$$(m \otimes h)(Y \xleftarrow{\beta} Z) := m \otimes \mathcal{H}(id_X, \beta)(h)$$

and

$$(W \xleftarrow{\alpha} X)(m \otimes h) := m_{(0)} \otimes \mathcal{H}(S^{-1}(m_{(-1)})(\alpha), id_Y)(h)$$

for any $m \otimes h$ from $M \ltimes \mathcal{H}(X, Y)$, $\alpha \in \text{Hom}_{\mathcal{C}}(W, X)$ and $\beta \in \text{Hom}_{\mathcal{C}}(Y, Z)$. For simplicity we will denote $\mathcal{H}(\alpha, \beta)(h)$ by $\alpha h \beta$ for any $h \in \mathcal{H}(X, Y)$, $\alpha \in \text{Hom}_{\mathcal{C}}(W, X)$ and $\beta \in \text{Hom}_{\mathcal{C}}(Y, Z)$.

Lemma 8.3. Let B , M , \mathcal{C} and \mathcal{H} be as in Definition 8.2. Then $M \ltimes \mathcal{H}$ is also a B -equivariant bifunctor on \mathcal{C} with values in $B\text{-}\mathbf{Mod}$.

Proof. For $b \in B$ and $m \otimes h$ in $M \ltimes \mathcal{H}(X, Y)$, and $\alpha \in \text{Hom}_{\mathcal{C}}(W, X)$ and $\text{Hom}_{\mathcal{C}}(Y, Z)$ we consider

$$b((m \otimes h)\beta) = b(m \otimes h\beta) = b_{(1)}(m) \otimes b_{(2)}(h\beta) = b_{(1)}(m) \otimes b_{(2)}(h)(b_{(3)}\beta) = b_{(1)}(m \otimes h)(b_{(2)}\beta)$$

and

$$\begin{aligned} b(\alpha(m \otimes h)) &= b(m_{(0)} \otimes (S^{-1}(m_{(-1)})\alpha)(h)) \\ &= b_{(1)}(m_{(0)}) \otimes (b_{(2)}S^{-1}(m_{(-1)})\alpha)b_{(3)}(h) \\ &= b_{(2)(2)}m_{(0)} \otimes (b_{(2)(3)}S^{-1}(m_{(-1)})S^{-1}(b_{(2)(1)})b_{(1)}\alpha)b_{(3)}(h) \\ &= (b_{(1)}\alpha)(b_{(2)}m \otimes b_{(3)}h) = (b_{(1)}\alpha)b_{(2)}(m \otimes h) \end{aligned}$$

which implies $M \ltimes \mathcal{H}$ is a B -equivariant bifunctor on \mathcal{C} with values in $B\text{-}\mathbf{Mod}$ whenever \mathcal{H} is a bifunctor on \mathcal{C} with values in $B\text{-}\mathbf{Mod}$ and M is a Yetter-Drinfeld module on B . \square

For a B -module algebra A and a Yetter–Drinfeld module M , we now define a new differential graded B -module $\mathrm{QCH}_*(A, B, A; M) := \mathrm{QCH}_*(*_B^A, B, M \ltimes \mathrm{Hom}_A(\cdot, \cdot))$ and obtain twisted version of the Morita equivalence

Theorem 8.4 (Morita invariance of twisted Hopf–Hochschild (co)homology). *Let B be a Hopf algebra and M be a Yetter–Drinfeld module over B . Assume also that A and A' are two B -module algebras. If the category of finitely generated B -equivariant representations of A and A' are B -equivariantly equivalent, then the twisted Hopf–Hochschild complexes ${}_B\mathrm{QCH}_*(A, B, A; M)$ and ${}_B\mathrm{QCH}_*(A', B, A'; M)$ are quasi-isomorphic.*

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